Simple analytic solution of fireball hydrodynamics

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A new family of simple analytic solutions of hydrodynamics is found for slowly expanding, rotationally symmetric fireballs assuming an ideal gas equation of state. The temperature profile is position independent only in the collisionless gas limit. The Zimányi-Bondorf-Garpman solution and the Buda-Lund parameterization of expanding hydrodynamic particle sources are recovered as special cases. The results are applied to predict new features of proton correlations and spectra at 1.93 AGeV $Ni + Ni$ collisions.

Introduction — Hydrodynamical models are applied in a wide range of physical problems from the evolution of galactic systems through the explosion of supernovas to the description of particle spectra and correlations in heavy ion collisions in a broad energy range. Due to the nonlinearity of the hydrodynamical equations it is rather difficult to solve them in an analytic manner. In particle and heavy ion physics some observables like the single-particle spectra and the two-particle Bose-Einstein correlation functions are sensitive to the means and variances of the hydrodynamical flow and density profiles [1,2]. This result implies that a hydrodynamical solution involving only the means and the variances of the density profile, flow profile and temperature profile is of great interest. Such a solution of the equations of non-relativistic (NR) hydrodynamics is presented herewith. Physical realization of the solution is possible both by heavy ion collisions in the 30 MeV A - 2 GeV A energy domain [3–5] and in a more limited sense by the NR transversal flows that seem to develop in high energy hadron-hadron and nucleus-nucleus collisions [1,6,7].

The new family of exact solutions — Let us consider the NR hydrodynamical problem, as specified by the continuity, Euler and energy equations:

$$\partial_t n + \nabla(\mathbf{v}n) = 0, \tag{1}$$

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = -(\nabla p)/(mn), \tag{2}$$

$$\partial_t \epsilon + \nabla(\epsilon \mathbf{v}) = -p\nabla\mathbf{v}, \tag{3}$$

where n denotes the number density of particles, $\mathbf{v}$ stands for the NR flow field, ϵ for the NR energy density, p for the pressure and in the following the temperature field is denoted by T. We assume a NR ideal gas equation of state,

$$p = nT, \tag{4}$$

$$\epsilon = \frac{3}{2}p = \frac{3}{2}nT, \tag{5}$$

which closes the set of equations for n, $\mathbf{v}$ and T.

We are looking for a set of solutions for a self-similar density, flow and temperature profile. Our ansatz for the self-similarity means that the non-trivial dependence of the solution on the spatial and temporal variables happens through the *scaling variable*

$$x = \mathbf{r}^2/R^2(t), \tag{6}$$

where $R(t)$ is a characteristic, time dependent radial scale. Note that the scaling variable x is not to be confused with r_x, the first component of the space vector $\mathbf{r} = (r_x, r_y, r_z)$. Let us assume rotational symmetry and introduce the dimensionless scaling functions $\nu(x)$, $\beta(x)$ and $\mathcal{T}(x)$ instead of $n = n(\mathbf{r}, t)$, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $T = T(\mathbf{r}, t)$, respectively:

$$n = n_0 \frac{R_0^3}{R^3}\, \nu(x), \tag{7}$$

$$\mathbf{v} = f_v(t)\, \beta(x)\, \mathbf{r}, \tag{8}$$

$$T = T_0\, f_T(t)\, \mathcal{T}(x), \tag{9}$$

the time-dependence of the radial scale is written as

$$R^2 = R^2(t) \,=\, R_0^2\, \phi(t), \tag{10}$$

and n_0, T_0 and R_0 are a constants related to the initial conditions. The form of this ansatz is not modified if we specify that the minimal "turning point radius" R_0 is prescribed at a given instant t_0 and the central particle density and temperature at this time t_0 is $n_0 = n(t_0, \mathbf{0})$ and $T_0 = T(t_0, \mathbf{0})$. This choice of constants implies

$$\nu(0) = \beta(0) \,=\, \mathcal{T}(0) \,=\, 1, \tag{11}$$

$$f_T(t_0) = \phi(t_0) = 1, \qquad R_0 \,=\, R(t_0). \tag{12}$$

The unknown functions $f_v(t)$, $f_T(t)$, $\phi(t)$ are determined as follows. From the continuity equation one obtains that

$$f_v(t) \,=\, \frac{R'}{R}, \qquad \text{and} \quad \beta(x) = 1, \tag{13}$$

where ordinary differentiation according to the argument is denoted by $'$. Thus the continuity equation implies

$$\mathbf{v} = \mathbf{r}\frac{R'}{R}, \tag{14}$$

regardless of the shape of the density profile.

Utilizing the equation of state and the continuity equation, the energy equation can be rewritten as

$$\partial_t T + (\mathbf{v}\nabla)T + \frac{2}{3}T\nabla\mathbf{v} \,=\, 0. \tag{15}$$

Inserting the ansatz into this form and making use of eq. (14) we obtain the time-dependence of the T as

$$f_T(t) \;=\; \frac{R_0^2}{R^2} \;=\; \frac{1}{\phi(t)}. \tag{16}$$

Inserting this to the Euler equation, we obtain

$$\phi''\phi - 0.5(\phi')^2 = \frac{2T_0}{mR_0^2} C_\phi, \tag{17}$$

$$\mathcal{T}' + \mathcal{T}\frac{\nu'}{\nu} = -\frac{C_\phi}{2}. \tag{18}$$

It is trivial to solve these equations for $\nu(x)$ and $\phi(t)$. The solution in terms of physical parameters reads as

$$\phi(t) = 1 + \frac{\langle u_t\rangle^2}{R_0^2}(t-t_0)^2, \qquad C_\phi \;=\; \frac{m}{T_0}\langle u_t\rangle^2, \tag{19}$$

$$\nu(x) = \frac{1}{\mathcal{T}(x)} \exp\left(-\frac{m\langle u_t\rangle^2}{2T_0}\int_0^x \frac{du}{\mathcal{T}(u)}\right). \tag{20}$$

The mean radial flow parameter $\langle u_t\rangle$ controls the *asymptotic* speed of the expansion, as $\lim_{t\to\infty} R'(t) = \langle u_t\rangle$. In the above solutions, the functional form of $\mathcal{T}(x)$, the scaling function of the radial temperature profile, can be chosen freely, corresponding to the freedom in the choice of the initial boundary conditions. These new hydro solutions read in terms of physical variables as :

$$R^2(t) = R_0^2 + \langle u_t\rangle^2 (t-t_0)^2, \tag{21}$$

$$\mathbf{v}(t,\mathbf{r}) = \frac{R'(t)}{R(t)}\,\mathbf{r} \;=\; \frac{\langle u_t\rangle^2 (t-t_0)}{R_0^2 + \langle u_t\rangle^2 (t-t_0)^2}\,\mathbf{r}, \tag{22}$$

$$T(t,\mathbf{r}) = T_0\,\mathcal{T}(x)\,\frac{R_0^2}{R^2} \;=\; T\left[t_0, \mathbf{r}R_0/R(t)\right]\,\frac{R_0^2}{R^2(t)} \tag{23}$$

$$n(t,\mathbf{r}) = n_0\,\frac{R_0^3}{R^3(t)}\,\frac{T(t_0,\mathbf{0})}{T\left[t_0,\mathbf{r}R_0/R(t)\right]} \;\times$$
$$\times\,\exp\left[-m\langle u_t\rangle^2 \frac{R^2(t)}{R_0^2}\int_0^{|\mathbf{r}|} r dr \frac{T(t_0,0)}{T(t_0,r)}\right]. \tag{24}$$

This solution describes a system that contracts for $t < t_0$, reaches its minimum size R_0 at the time of the turning point $t = t_0$ when the flow field vanishes. For later times, $t > t_0$, the system expands again. The temperature field is obtained by rescaling the initial temperature field.

The above explicit solution for the density profile can be written into a simpler form using the scaling variable x and the scaling function of the temperature profile, $\mathcal{T}(x)$:

$$n(t,\mathbf{r}) \;=\; \frac{n_0}{\mathcal{T}(x)}\,\frac{R_0^3}{R^3(t)}\,\exp\left[-\frac{m\langle u_t\rangle^2}{2T_0}\int_0^x \frac{du}{\mathcal{T}(u)}\right]. \tag{25}$$

This completes the generic description of our solution of NR spherically symmetric fireball hydrodynamics. The solution, given by eqs. (21-25), depends on 4 constants of integration, R_0, $\langle u_t\rangle$, T_0 and t_0, as well as on the dimensionless *function* $\mathcal{T}(x)$. Three of the constants of integration correspond to the three integrated equations (continuity, Euler and energy equation). The constant t_0 corresponds to the homogeneity of the equations of NR hydrodynamics in time. The free choice of the dimensionless scaling function corresponds to the freedom in the specification of the initial boundary conditions. Thus we have found infinitely many new solutions of NR hydrodynamics, one for each integrable function of one variable. Note that each of these solutions is scale invariant:

$$n\left(t, \mathbf{r}\frac{R}{R_0}\right)\,\frac{R^3}{R_0^3} = n(t_0,\mathbf{r}), \tag{26}$$

$$\mathbf{v}\left(t, \mathbf{r}\frac{R}{R_0}\right)\,\frac{R_0'}{R'} = \mathbf{v}(t_0,\mathbf{r}), \tag{27}$$

$$T\left(t, \mathbf{r}\frac{R}{R_0}\right)\,\frac{R^2}{R_0^2} = T(t_0,\mathbf{r}), \tag{28}$$

independently of the time t, where $R_0' = R'(t_0)$.

The self-similarity can be utilized to match this solution to any given initial time t_i for a given initial radius R_i and a given (rotationally symmetric) temperature distribution $T_i(\mathbf{r})$ at time t_i. These initial conditions are

$$T_i(\mathbf{r}) = T(t_i,\mathbf{r}), \quad R(t_i) \;=\; R_i, \tag{29}$$

$$\mathbf{v}(\mathbf{r},\, t_i) = \frac{\mathbf{r}}{R_i}\langle u_i\rangle, \quad n(t_i,\mathbf{0}) \;=\; n_i. \tag{30}$$

This yields the following solutions for ϕ and $R(t)$:

$$\phi_i(t) = \left(1 + \frac{\langle u_i\rangle}{R_i}(t-t_i)\right)^2 + \frac{\langle u_t\rangle^2}{R_i^2}(t-t_i)^2, \tag{31}$$

$$R(t)^2 = R_i^2\phi_i(t), \qquad T_i \;=\; T_i(\mathbf{0}). \tag{32}$$

This solution appears instead of eqs. (19, 21) if the initial temperature field, initial radius and initial flow field is prescribed instead of the turning point temperature and radius.

The new solution presented above also solves the equation of state of an ideal gas for isentropic expansion [8]:

$$\left[\partial_t + (\mathbf{v}\nabla)\right] p n^{-5/3} \;=\; 0. \tag{33}$$

Note that *the time-dependence of the radius and the solution for the flow-field is independent of the initial temperature profile* $T(t_0,\mathbf{r})$. Due to this property of our general solution , we can study various initial conditions in a simple manner, assuming various distributions of $T(t_0,\mathbf{r})$ and evaluate its influence on the density profile only.

Knudsen-gas limit — An essential feature of our solution is that the central and the asymptotic temperatures may differ and induce a radial dependence of the temperature field. If the initial temperature field is constant in space, we find a special case of $\mathcal{T}(x) = 1$ and $T(t_0,\mathbf{r}) = T(t_0) = T_0$. Eq. (25) yields that the density profile is Gaussian with $R_G^2 = R^2 T_0/(m\langle u_t\rangle^2)$:

$$n = \frac{N}{[2\pi R_G^2(t)]^{3/2}}\exp\left(-\frac{\mathbf{r}^2}{2R_G^2(t)}\right). \tag{34}$$

It is straightforward to prove [9] by evaluating the time dependence of the phase-space density, that this case corresponds to a collisionless, free streaming of the particles. Hence this parameterization can be applied only to describe the post-freeze-out stage of heavy ion collisions. Thus, *non-trivial hydrodynamical evolution takes place in the presence of a non-vanishing temperature gradient only*, for all rotationally symmetric solutions of the non-relativistic hydrodynamical equations that we found.

The Zimányi-Bondorf-Garpman solution — 20 years ago, a famous analytic solution of NR hydrodynamics was found by Zimányi, Bondorf and Garpman (ZBG) in ref. [8]. The temperature and the density for that solution explicitly vanishes if $|\mathbf{r}|$ is larger than an expanding radial scale, $|\mathbf{r}| > R(t)$. The ZBG solution has been extensively applied to describe particle spectra for NR heavy ion collisions. The ZBG solution has the following simple initial temperature profile:

$$T(t_0,\mathbf{r}) = T_0 \left[1 - \frac{\mathbf{r}^2}{R^2(t)}\right] \qquad \text{if } |\mathbf{r}| \leq R(t). \tag{35}$$

Inserting this to our general solution, eq. (25), we obtain

$$n(t,\mathbf{r}) = n_0 \frac{R_0^3}{R^3} \left[1 - \frac{\mathbf{r}^2}{R^2}\right]^{C_\phi/2-1} \qquad \text{if } |\mathbf{r}| \leq R(t), \tag{36}$$

which corresponds to the ZBG solution with an exponent $\alpha = C_\phi/2 - 1 = \frac{m\langle u_t\rangle^2}{2T_0} - 1$.

Generalization to d-dimensional expansion – Our analytic solution can be generalized to expansions in d dimensions, where d is an arbitrary number. The case $d = 2$ is especially interesting as it corresponds to the time evolution of hot and dense hadronic matter in the transverse directions in high energy heavy ion [1,10,11] and in particle reactions [6]. When analyzing particle correlations and spectra in these reactions, it is found that the longitudinal expansion is relativistic, (sometimes extremely relativistic), however, this longitudinal flow can be approximately decoupled from the transversal dynamics, which is found to be non-relativistic [11,1].

In d dimensions, the divergences of the flow and the equation of state are modified as

$$\nabla\mathbf{r}_d \;=\; d, \qquad \epsilon_d \;=\; \frac{d}{2}p. \tag{37}$$

The scaling variable is $x = \mathbf{r}_d^2/R^2(t)$, the ansatz is

$$n \;=\; n_0 \frac{R_0^d}{R^d(t)}\nu(x) \quad \mathbf{v}_d \;=\; \mathbf{r}_d f_d(t)\beta(x). \tag{38}$$

The equations for the temperature and the radial scale parameter are unchanged. In d dimensions, the energy equation eq. (15) is modified to

$$\partial_t T + (\mathbf{v}_d\nabla)T + \frac{2}{d}T\nabla\mathbf{v}_d \;=\; 0. \tag{39}$$

From this and the continuity equation we obtain after eliminating the singular solutions that

$$\mathbf{v}_d = \mathbf{r}_d \frac{R'(t)}{R(t)} \tag{40}$$

and the dimension of the expansion, d, cancels from the Euler equation. The solutions for d dimensional expansions are very similar to the $d = 3$ dimensional expanding fireballs, the differences are that in eqs. (22,23) one has to utilize d dimensional vectors for coordinate-space and flow, and in eq. (24) the density decreases as $n \propto R(t)^{-d}$.

The Buda-Lund parameterization — The Buda-Lund parameterization (BL) was introduced to describe phenomenologically the NR transversal flows appearing in high energy hadron-hadron and heavy ion reactions, given first in refs. [1,10]. This parameterization keeps only the means and the variances of the density distribution, the inverse temperature profile and it corresponds to a scaling transversal flow, such as eq. (22). In refs. [1,4,6,11], a radial temperature profile was used,that can be rewritten as:

$$T(\tau,\mathbf{r}) = \frac{T_0}{\left[1+\langle\frac{\Delta T}{T}\rangle_r \frac{\mathbf{r}^2}{2R^2}\right]\left[1+\langle\frac{\Delta T}{T}\rangle_\tau \frac{(\tau-\tau_0)^2}{2\Delta\tau^2}\right]}, \tag{41}$$

where $\mathbf{r} = (r_x, r_y)$, the change of the temperature in the radial and the temporal directions is controlled by the free parameters $\langle\Delta T/T\rangle_r$ and $\langle\Delta T/T\rangle_\tau$, respectively, and C is a constant of normalization. In the above parameterization, the longitudinal proper-time, $\tau = \sqrt{t^2 - r_z^2}$ is introduced instead of the time t , corresponding to a relativistic longitudinal expansion. The exact Buda-Lund density profile is obtained from eqs. (25,41) as

$$n(t_0,\mathbf{r}) = n_0 \left(1 + \langle\Delta T/T\rangle_r \mathbf{r}^2/(2R_0^2)\right) \times \exp\left[-\frac{m\langle u_t\rangle^2}{2T_0}\left(\frac{\mathbf{r}^2}{R_0^2} + \langle\frac{\Delta T}{T}\rangle_r \frac{\mathbf{r}^4}{R_G^4}\right)\right]. \tag{42}$$

The time-evolution of this density can be obtained from simple re-scaling, eq. (26), and comparing the BL ansatz for the time dependence of the cooling, eq. (41) with the exact hydro solution we find

$$\frac{1}{2\Delta\tau^2}\langle\Delta T/T\rangle_\tau \;=\; \frac{\langle u_t\rangle^2}{R_0^2}. \tag{43}$$

The exact density distribution in eq. (42) can be approximated by a Gaussian, corresponding to the BL Gaussian ansatz for the density profile in ref. [1], if terms of $\mathcal{O}(\mathbf{r}^4/R^4)$ can be neglected. (Note that the difference between the BL and the ZBG solutions is only $\mathcal{O}(\mathbf{r}^4/R^4)$ if one sets $\langle\Delta T/T\rangle_r = 1$ in the BL hydro solution.) The BL Gaussian radius is

$$\frac{1}{R_G^2} \;=\; \frac{1}{R_0^2}\left(\frac{m\langle u_t\rangle^2}{T_0} - \langle\Delta T/T\rangle_r\right) \geq 0. \tag{44}$$

Hence the BL parameterization is found to correspond to a Gaussian approximation to an exact solution of the non-relativistic hydrodynamical equations, with constrained values of the cooling parameters $\langle \Delta T/T\rangle_\tau$ and $\langle \Delta T/T\rangle_r$. If the inequality in eq. (44) is violated, the solution for the density profile in eq. (42) will be still valid. In this case, the radial density profile will increase first before decreasing at large distances. This type of solution corresponds to a kind of *expanding "smoke-ring"* in the transversal plane, or, to an *expanding spherical shell* if the expansion is three dimensional, as illustrated on Figure 1 for $R_0 = 5$ fm, $m\langle u_t\rangle^2/T_0 = 0.125$. For $\langle \Delta T/T\rangle_r = 0.06 < m\langle u_t\rangle^2/T_0$ the density has one maximum while for $\langle \Delta T/T\rangle_r = 0.2 > m\langle u_t\rangle^2/T_0$ the density profile develops two maxima in any transversal direction.

Application to FOPI data — The FOPI Collaboration measured recently the proton-proton correlation functions at 1.93 AGeV Ni + Ni collisions [5]. To interpret their data, they utilized, independently, a version of the hydrodynamical solution, presented here. They assumed a linear flow profile, a Gaussian density distribution and a constant temperature. This solution corresponds to a collisionless Knudsen gas [9]. One expects that a collisionless approximation breaks down for 1.93 AGeV Ni + Ni collisions. Indeed, only the peak of the FOPI experimental proton-proton correlation functions was well reproduced by the collisionless model, however, the tails had to be excluded from the fit. From the above presented solution of the equations of the NR hydrodynamical systems we know, that the temperature distribution must have a radial and a temporal profile *if* a non-trivial hydro evolution is expected. Such a situation was also studied in ref. [4], where it was shown that a temperature profile results in a momentum dependence of the effective radius parameters of the proton-proton correlation functions and in a characteristic bending down of the proton- spectra, observable as a deviation from the exponential spectrum [5]. Hence we predict a deviation from the exponential proton spectrum and a momentum dependence of the effective source radii, to be observed in a more detailed analysis of the FOPI data. Note, that comparison with experimental data with the help of a hydro solution requires also a condition for the particle freeze-out, see ref. [8] for details.

Summary — A new family of simple and exact solutions of the non-relativistic hydrodynamical equations is found for rotationally symmetric fireballs, for an ideal gas equation of state. Without radial temperature gradients, the only solution of these equations was found to be a collisionless Knudsen gas lacking dynamical evolution. The results are generalized to expansion in arbitrary number of spatial dimensions. The Zimányi-Bondorf-Garpman solution [8] and the Buda-Lund hydro parameterization [1] are recovered as special cases. For strong enough radial decrease of the temperature, expanding smoke-ring type of hydro solutions are described.

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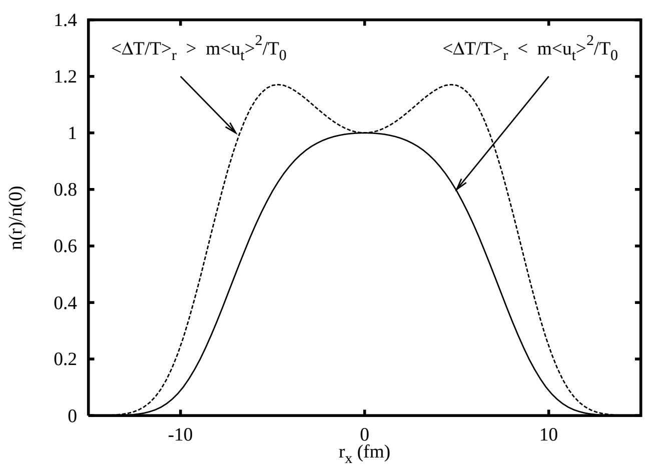

FIG. 1. Illustration of the development of smoke-ring solutions for large temperature gradients in exact solutions of non-relativistic hydrodynamics.